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DYNAMICAL ASPECTS OF THE DEFORMATION OF A 6-CONSTANT CUBIC CRYSTAL

by

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ABSTRACT

The displacement equations of motion for a 6-constant centro-symmetric cubic crystalline material are developed. Also, the determinantal equations for velocities of plane elastic waves are developed. Previously determined elastic constants are used to calculate the phase velocities in various directions of a cubic crystal. Normal mode vibrations of cubes and infinite prisms are theoretically considered. Dilatational modes of these bodies are determined and they demonstrate that there are no couple-stress effects in these cases. Normal mode shapes and frequencies for the equivoluminal modes are obtained but without consideration of couple-stresses.

Free vibration experiments were conducted on a two-dimensional model which was considered as a slice of an infinite prism and on a three-dimensional cubic model. Experimentally determined frequencies and mode shapes are studied in terms of the theoretical results.

INTRODUCTION

In a previous study [1][1], a physical model of a so-called 6-constant centro-symmetric cubic crystal was introduced in order to illustrate the phenomenon of couple-stresses and to determine the numerical values of the elastic constants. As concluded from that work the constants were satisfactorily determined, but it was then considered worthwhile to perform normal mode vibration experiments to further explore the theory and examine its completeness. In addition, it is thought to be useful to investigate the nature of elastic wave propagation in such materials. For these purposes the present research has been conducted. Consequently the equations of motion for the models will be developed and solved for the cases of interest.

EQUATIONS OF MOTION

The three-dimensional stress equations of motion in terms of force-stress and couple-stress may be written down directly from the equations of equilibrium which were previously given [1]:

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[1] Numbers in brackets refer to references at end of report.

$$\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = \rho\,\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} = \rho\,\frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z} = \rho\,\frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + \tau_{yz} - \tau_{zy} = 0 \qquad (1)$$

$$\frac{\partial q_{xy}}{\partial x} + \frac{\partial q_{yy}}{\partial y} + \frac{\partial q_{zy}}{\partial z} + \tau_{zx} - \tau_{xz} = 0$$

$$\frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial q_{zz}}{\partial z} + \tau_{xy} - \tau_{yx} = 0$$

On account of nonsymmetry of the force-stress tensor we may write it as,

$$\tau_{ij} = \tau^s_{ij} + \tau^a_{ij} \qquad (2)$$

where

$$\tau^s_{ij} = \frac{1}{2}(\tau_{ij} + \tau_{ji})$$

$$\tau^a_{ij} = \frac{1}{2}(\tau_{ij} - \tau_{ji})$$

The couple-stress tensor may be written as sum of deviatoric and spherical parts as follows:

$$q_{ij} = q_{ij}^d + \frac{1}{3} q_{kk} \delta_{ij} \tag{3}$$

where q_{ij}^d stands for deviatoric part and repeated index means summation. Now the constitutive equations for a six-constant centrosymmetric cubic material may be written as follows:

$$\begin{aligned}
\tau_{xx}^s &= S_{11} \epsilon_{xx} + S_{12} \epsilon_{yy} + S_{12} \epsilon_{zz} \\
\tau_{yy}^s &= S_{12} \epsilon_{xx} + S_{11} \epsilon_{yy} + S_{12} \epsilon_{zz} \\
\tau_{zz}^s &= S_{12} \epsilon_{xx} + S_{12} \epsilon_{yy} + S_{11} \epsilon_{zz} \\
\tau_{yz}^s &= S_{44} \epsilon_{yz} \\
\tau_{xz}^s &= S_{44} \epsilon_{xz} \\
\tau_{xy}^s &= S_{44} \epsilon_{xy}
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&q_{xx}^d = \frac{3}{2} A\, k_{xx}, \quad q_{yy}^d = \frac{3}{2} A\, k_{yy}, \quad q_{zz}^d = \frac{3}{2} A\, k_{zz} \\
&q_{yz}^d = C\, k_{yz} + B\, k_{zy}, \quad q_{zy}^d = C\, k_{zy} + B\, k_{yz} \\
&q_{xz}^d = C\, k_{xz} + B\, k_{zx}, \quad q_{zx}^d = C\, k_{zx} + B\, k_{xz} \\
&q_{xy}^d = C\, k_{xy} + B\, k_{yx}, \quad q_{yx}^d = C\, k_{yx} + B\, k_{xy}
\end{aligned} \tag{5}$$

In the case of isotropy, the following relations hold:

$$S_{44} = S_{11} - S_{12}$$

and

$$\frac{3A}{2} = B + C$$

The strains and curvatures may be expressed in terms of displacement functions u, v, w as follows:

$$\begin{aligned}
&\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} \\
&\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\
&\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&k_{xx} = \frac{\partial \omega_x}{\partial x}, \quad k_{xy} = \frac{\partial \omega_y}{\partial x}, \quad k_{xz} = \frac{\partial \omega_z}{\partial x} \\
&k_{yx} = \frac{\partial \omega_x}{\partial y}, \quad k_{yy} = \frac{\partial \omega_y}{\partial y}, \quad k_{yz} = \frac{\partial \omega_z}{\partial y} \\
&k_{zx} = \frac{\partial \omega_x}{\partial z}, \quad k_{zy} = \frac{\partial \omega_y}{\partial z}, \quad k_{zz} = \frac{\partial \omega_z}{\partial z}
\end{aligned} \tag{7}$$

where the ω's are the rotations whose components are:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

On substitution of equations (4), (5), (6) and (7) into equation (1) and after proper grouping the displacement equations of motion are as follows:

$$S_{11}\frac{\partial^2 u}{\partial x^2} + S_{12}\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right) + \frac{S_{44}}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right)$$

$$+ \frac{C}{4}\left(\frac{\partial^4 v}{\partial y\partial x^3} + \frac{\partial^4 v}{\partial x\partial y^3} + \frac{\partial^4 w}{\partial z\partial x^3} + \frac{\partial^4 w}{\partial z^3\partial x} - \frac{\partial^4 u}{\partial y^2\partial x^2} - \frac{\partial^4 u}{\partial y^4} - \frac{\partial^4 u}{\partial z^2\partial x^2} - \frac{\partial^4 u}{\partial z^4}\right)$$

$$+ \frac{B}{4}\left(2\frac{\partial^4 u}{\partial y^2\partial z^2} - \frac{\partial^4 v}{\partial x\partial y\partial z^2} - \frac{\partial^4 w}{\partial x\partial z\partial y^2}\right)$$

$$+ \frac{3A}{8}\left(\frac{\partial^4 v}{\partial x\partial y\partial z^2} + \frac{\partial^4 w}{\partial x\partial z\partial y^2} - 2\frac{\partial^4 u}{\partial y^2\partial z^2}\right) = \rho\frac{\partial^2 u}{\partial t^2}$$

$$S_{11}\frac{\partial^2 v}{\partial y^2} + S_{12}\left(\frac{\partial^2 w}{\partial y\partial z} + \frac{\partial^2 u}{\partial y\partial x}\right) + \frac{S_{44}}{2}\left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y\partial z} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y\partial x}\right)$$

$$+ \frac{C}{4}\left(\frac{\partial^4 w}{\partial y^3\partial z} + \frac{\partial^4 w}{\partial z^3\partial y} + \frac{\partial^4 u}{\partial y^3\partial x} + \frac{\partial^4 u}{\partial y\partial x^3} - \frac{\partial^4 v}{\partial y^2\partial z^2} - \frac{\partial^4 v}{\partial y^2\partial x^2} - \frac{\partial^4 v}{\partial z^4} - \frac{\partial^4 v}{\partial x^4}\right)$$

$$+ \frac{B}{4}\left(2\frac{\partial^4 v}{\partial z^2\partial x^2} - \frac{\partial^4 w}{\partial y\partial z\partial x^2} - \frac{\partial^4 u}{\partial y\partial z^2\partial x}\right)$$

$$+ \frac{3A}{8}\left(\frac{\partial^4 w}{\partial y\partial z\partial x^2} - 2\frac{\partial^4 v}{\partial z^2\partial x^2} + \frac{\partial^4 u}{\partial y\partial z^2\partial x}\right) = \rho\frac{\partial^2 v}{\partial t^2}$$

$$S_{11}\frac{\partial^2 w}{\partial z^2} + S_{12}\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 v}{\partial z\partial y}\right) + \frac{S_{44}}{2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial z\partial y}\right)$$

$$+ \frac{C}{4}\left(\frac{\partial^4 u}{\partial z^3\partial x} + \frac{\partial^4 u}{\partial x^3\partial z} + \frac{\partial^4 v}{\partial z^3\partial y} + \frac{\partial^4 v}{\partial z\partial y^3} - \frac{\partial^4 w}{\partial z^2\partial x^2} - \frac{\partial^4 w}{\partial z^2\partial y^2} - \frac{\partial^4 w}{\partial x^4} - \frac{\partial^4 w}{\partial y^4}\right)$$

$$+ \frac{B}{4}\left(2\frac{\partial^4 w}{\partial x^2\partial y^2} - \frac{\partial^4 u}{\partial z\partial x\partial y^2} - \frac{\partial^4 v}{\partial z\partial x^2\partial y}\right)$$

$$+ \frac{3A}{8}\left(\frac{\partial^4 u}{\partial z\partial x\partial y^2} - 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 v}{\partial z\partial x^2\partial y}\right) = \rho\frac{\partial^2 w}{\partial t^2} \qquad (8)$$

PROPAGATION OF ELASTIC WAVES

The differential equations of motion (8) can now be used to study the propagation of elastic waves. The displacements for plane waves in an infinite elastic media may be assumed to be of the usual form [2],

$$(u,v,w) = (U,V,W)e^{i\frac{2\pi}{\lambda}(\ell x + my + nz - Ct)} \tag{9}$$

where U, V, W are the components of displacement amplitude, λ is the wave length, C is the velocity of propagation, (ℓ, m, n) are direction cosines of the normal to wave front.

On substitution of the displacement functions (9) into equations of motion (8), we obtain the following three equations in terms of the amplitude components U, V, W:

$$\begin{aligned}
(\Gamma_{11} - \rho C^2)U + \Gamma_{12}V + \Gamma_{13}W &= 0 \\
\Gamma_{21}U + (\Gamma_{22} - \rho C^2)V + \Gamma_{23}W &= 0 \\
\Gamma_{31}U + \Gamma_{32}V + (\Gamma_{33} - \rho C^2)W &= 0
\end{aligned} \tag{10}$$

where

$$\Gamma_{11} = S_{11}\ell^2 + \frac{S_{44}}{2}(m^2 + n^2) + \frac{C}{4}k^2(m^2\ell^2 + m^4 + n^2\ell^2 + n^4)$$

$$- \frac{2B}{4}k^2m^2n^2 + \frac{6A}{8}k^2m^2n^2$$

$$\Gamma_{22} = S_{11}m^2 + \frac{S_{44}}{2}(n^2 + \ell^2) + \frac{C}{4}k^2(n^2m^2 + n^4 + \ell^2m^2 + \ell^4)$$

$$- \frac{2B}{4}k^2n^2\ell^2 + \frac{6A}{8}k^2n^2\ell^2$$

$$\Gamma_{33} = S_{11}n^2 + \frac{S_{44}}{2}(\ell^2 + m^2) + \frac{C}{4}k^2(\ell^2n^2 + \ell^4 + m^2n^2 + m^4)$$

$$- \frac{2B}{4}k^2\ell^2m^2 + \frac{6A}{8}k^2\ell^2m^2$$

$$\Gamma_{12} = (S_{12} + \frac{S_{44}}{2})\ell m - \frac{C}{4}k^2(\ell^3m + \ell m^3) + \frac{B}{4}k^2\ell mn^2 - \frac{3A}{8}k^2\ell mn^2$$

$$\Gamma_{23} = (S_{12} + \frac{S_{44}}{2})mn - \frac{C}{4}k^2(m^3n + mn^3) + \frac{B}{4}k^2mn\ell^2 - \frac{3A}{8}k^2mn\ell^2$$

$$\Gamma_{13} = (S_{12} + \frac{S_{44}}{2})\ell n - \frac{C}{4}k^2(n\ell^3 + n^3\ell) + \frac{B}{4}k^2\ell nm^2 - \frac{3A}{8}k^2\ell nm^2 \qquad (11)$$

$$k = \frac{2\pi}{\lambda}$$

The condition for nontrivial solutions of equations (10) is that the determinant of coefficients should vanish,

$$\begin{vmatrix} (\Gamma_{11} - \rho C^2) & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & (\Gamma_{22} - \rho C^2) & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & (\Gamma_{33} - \rho C^2) \end{vmatrix} = 0 \qquad (12)$$

Solution of equation (12) for an arbitrary direction (ℓ, m, n) of the normal to wave front and for nonzero values of the amplitude components U, V and W is very laborious and unnecessary for present purposes. In order to investigate the influence of couple-stress on the wave velocities and to identify the associated nature of deformation plane wave normals may be chosen in the directions shown in Figure 1. These directions are those of the cubic axis [100], the face diagonal [110] and the body diagonal [111]. By appropriate selection of U, V and W the phase velocities can be determined as definite functions of the elastic constants. The phase velocities in each of the above mentioned directions are separately treated for the following cases.

Case (1): Wave Normal Parallel to Cubic Axis [100]

In this direction,

$$\ell = 1, \quad m = n = 0$$

If we take all the displacement components as nonzero, then there will be three velocities in each direction.

However, if we consider

$$U \neq 0$$
$$V \neq 0$$
$$W = 0$$

then equation (12) will reduce to the following:

$$(\Gamma_{11} - \rho C^2)(\Gamma_{22} - \rho C^2) - \Gamma_{12}^2 = 0 \tag{13}$$

from which there results the following relations,

$$\rho C_1{}^2 = S_{11} \tag{14}$$

and

$$\rho C_2{}^2 = \frac{S_{44}}{2} + \frac{C}{4} k^2$$

where C_1 and C_2 are the phase velocities. Also, another form of displacement amplitudes which will yield a simple velocity equation is,

$$U = V = 0$$
$$W \neq 0$$

to give

$$\rho C_3{}^2 = \frac{S_{44}}{2} + \frac{C}{4} k^2 \tag{15}$$

where C_3 is the phase velocity. It may be noticed from equations (14) and (15) that $C_2 = C_3$.

Case (2): Wave Normal Parallel to Face Diagonal [110]

In this direction,

$$\ell = m = \frac{1}{\sqrt{2}}, \qquad n = 0$$

and for

$$U \neq 0$$
$$V \neq 0$$
$$W = 0$$

the velocities are given by:

$$\rho C_1^{\,2} = \frac{S_{11} + S_{12} + S_{44}}{2}$$

$$\rho C_2^{\,2} = \frac{1}{2}\left(S_{11} - S_{12} + \frac{C}{2} k^2\right) \tag{16}$$

and also for

$$U = V = 0$$
$$W \neq 0$$

the velocity is given by

$$\rho C_3^{\,2} = \frac{S_{44}}{2} + \frac{C}{8} k^2 - \frac{B}{8} k^2 + \frac{3A}{16} k^2 \tag{17}$$

Case (3): Wave Normal Parallel to Body Diagonal [111]

In this direction,

$$\ell = m = n = \frac{1}{\sqrt{3}}$$

and for

$$U = V \neq 0$$

$$W \neq 0$$

the wave velocities are given by:

$$\rho C_1{}^2 = \frac{S_{11}}{3} + \frac{2}{3} S_{12} + \frac{2}{3} S_{44} - \frac{B}{9} k^2 + \frac{A}{6} k^2$$

$$\rho C_2{}^2 = \frac{S_{11}}{3} - \frac{S_{12}}{3} + \frac{S_{44}}{6} + \frac{C}{18} k^2 - \frac{5B}{36} k^2 + \frac{5A}{24} k^2 \qquad (18)$$

and also for

$$U = -V$$

and

$$W = 0$$

the wave velocity is given by

$$\rho C_3{}^2 = \frac{S_{11}}{3} - \frac{S_{12}}{3} + \frac{S_{44}}{6} + \frac{C}{18} k^2 - \frac{5B}{36} k^2 + \frac{5A}{24} k^2 \qquad (19)$$

Note that $C_2 = C_3$ in this case.

Density of the model is assumed to be the ratio of total mass of the unit-cell to the volume of unit-cell. The value for the model under consideration is, $\rho = \frac{.325 \times 10^{-4}}{h} \; \frac{\text{lbs.sec.}^2}{\text{in.}^4}$, where h (3 inches) is the center to center distance between the blocks. For the case of $\lambda = h$, the wave velocities are tabulated in Table I. If the couple-stress effect is neglected all the velocity expressions reduce to those of the classical case for a cubic crystal [2].

NORMAL MODE VIBRATIONS

Now consideration will be given to the normal mode vibrations of the crystals. The vibrating body is assumed to have stress-free boundary conditions. Theoretical studies of normal mode vibrations of an infinite prism on the basis of plane strain equations and of a cube on the basis of three-dimensional equations are considered.

1. Plane Strain Vibrations of Infinite Prisms

The displacement equations of motion (8) for plane strain reduce to,

$$S_{11}\frac{\partial^2 u}{\partial x^2} + S_{12}\frac{\partial^2 v}{\partial x\partial y} + \frac{S_{44}}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}\right.$$

$$\left. + \frac{C}{4}\left(\frac{\partial^4 v}{\partial x^3\partial y} + \frac{\partial^4 v}{\partial y^3\partial x} - \frac{\partial^4 u}{\partial x^2\partial y^2} - \frac{\partial^4 u}{\partial y^4}\right) = \rho\frac{\partial^2 u}{\partial t^2}\right.$$

$$S_{11}\frac{\partial^2 v}{\partial y^2} + S_{12}\frac{\partial^2 u}{\partial y\partial x} + \frac{S_{44}}{2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y\partial x}\right)$$

$$+ \frac{C}{4}\left(\frac{\partial^4 u}{\partial y^3\partial x} + \frac{\partial^4 u}{\partial y\partial x^3} - \frac{\partial^4 v}{\partial y^2\partial x^2} - \frac{\partial^4 v}{\partial x^4}\right) = \rho\frac{\partial^2 v}{\partial t^2} \qquad (20)$$

Taking the origin at the center of the model, we may write down the stress-free surface conditions which should be satisfied at $x = \pm a$ and $y = \pm a$ as follows:

$$\tau^s_{xx}\ell + \tau^s_{yx}m + \frac{1}{2}\left(\frac{\partial q^d_{xz}}{\partial x} + \frac{\partial q^d_{yz}}{\partial y} + \frac{\partial q^d_{zz}}{\partial z} - \frac{\partial \overline{Q}^d}{\partial z}\right)m = 0$$

$$\tau^s_{xy}\ell + \tau^s_{yy}m - \frac{1}{2}\left(\frac{\partial q^d_{xz}}{\partial x} + \frac{\partial q^d_{yz}}{\partial y} + \frac{\partial q^d_{zz}}{\partial z} - \frac{\partial \overline{Q}^d}{\partial z}\right)\ell = 0$$

$$\tau^s_{xz}\ell + \tau^s_{yz}m + \frac{1}{2}\left(\frac{\partial q^d_{xy}}{\partial x} + \frac{\partial q^d_{yy}}{\partial y} + \frac{\partial q^d_{zy}}{\partial z} - \frac{\partial \overline{Q}^d}{\partial y}\right)\ell$$

$$- \frac{1}{2}\left(\frac{\partial q^d_{xx}}{\partial x} + \frac{\partial q^d_{yx}}{\partial y} + \frac{\partial q^d_{zx}}{\partial z} - \frac{\partial \overline{Q}^d}{\partial x}\right)m = 0$$

$$q^d_{xx}\ell + q^d_{yx}m - \overline{Q}^d\ell = 0$$

$$q^d_{xy}\ell + q^d_{yy}m - \overline{Q}^d m = 0$$

$$q^d_{xz}\ell + q^d_{yz}m = 0 \tag{21}$$

where ℓ, m, n are direction cosines of unit normal to the surface

$$\overline{Q}^d = q^d_{xx}\ell^2 + q^d_{yy}m^2 + (q^d_{xy} + q^d_{yx})\ell m$$

(a) A Dilatational Mode

Since the dilatational mode consists only of extension and contraction of principal line elements of the material, the frequencies are not influenced by couple-stress. Displacement functions which will satisfy equations (20) and (21) for the case of vanishing Poisson's ratio are found to be,

$$u = A \sin(\frac{m\pi}{2a})x \, e^{i\omega t}$$
$$v = A \sin(\frac{m\pi}{2a})y \, e^{i\omega t} \qquad (22)$$

with m representing the odd integers. The above functions will describe symmetric modal shapes with respect to the coordinate axes. Asymmetric modes are given by

$$u = A \cos(\frac{m\pi}{2a})x \, e^{i\omega t}$$
$$v = A \cos(\frac{m\pi}{2a})y \, e^{i\omega t} \qquad (23)$$

with m representing the even integers.

The frequency equation corresponding to functions (22) and (23) are found to be,

$$\rho\omega^2 = S_{11}(\frac{m\pi}{2a})^2 \qquad (24)$$

where $m = 1, 2, 3, \ldots$.

(b) An Equivoluminal Mode Without Couple-stresses

Without consideration of the couple-stresses, the differential equations of motion (20) and boundary conditions (21) reduce to

$$S_{11}\frac{\partial^2 u}{\partial x^2} + S_{12}\frac{\partial^2 v}{\partial x \partial y} + \frac{S_{44}}{2}(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}) = \rho\frac{\partial^2 u}{\partial t^2}$$
$$S_{11}\frac{\partial^2 v}{\partial y^2} + S_{12}\frac{\partial^2 u}{\partial y \partial x} + \frac{S_{44}}{2}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x}) = \rho\frac{\partial^2 v}{\partial t^2} \qquad (25)$$

and

$$\begin{aligned}\tau^s_{xx}\ell + \tau^s_{yx}m &= 0\\ \tau^s_{xy}\ell + \tau^s_{yy}m &= 0\\ \tau^s_{xz}\ell + \tau^s_{yz}m &= 0\end{aligned} \tag{26}$$

respectively.

Displacement functions which will satisfy equations (25) and (26) for symmetric modes are seen to be the following:

$$\begin{aligned}u &= A \sin\left(\frac{m\pi}{2a}\right)x \cos\left(\frac{m\pi}{2a}\right)y\ e^{i\omega t}\\ v &= -A \cos\left(\frac{m\pi}{2a}\right)x \sin\left(\frac{m\pi}{2a}\right)y\ e^{i\omega t}\end{aligned} \tag{27}$$

with m representing the odd integers. The asymmetric modes are given by,

$$\begin{aligned}u &= A \cos\left(\frac{m\pi}{2a}\right)x \sin\left(\frac{m\pi}{2a}\right)y\ e^{i\omega t}\\ v &= -A \sin\left(\frac{m\pi}{2a}\right)x \cos\left(\frac{m\pi}{2a}\right)y\ e^{i\omega t}\end{aligned} \tag{28}$$

with m representing the even integers.

The frequency equation corresponding to functions (27) and (28) are found to be,

$$\rho\omega^2 = (S_{11} - S_{12})\left(\frac{m\pi}{2a}\right)^2 \tag{29}$$

2. Vibration of a Cube

All three displacement functions u, v, w are involved in this case and the equations of motion are equations (8). Considering the center of the cube as origin of the coordinate system, the conditions of stress-free surfaces at $x = \pm a$, $y = \pm a$ and $z = \pm a$ are the following:

$$\tau_{xx}^{s}\ell + \tau_{yx}^{s}m + \tau_{zx}^{s}n + \frac{1}{2}\left(\frac{\partial q_{xz}^{d}}{\partial x} + \frac{\partial q_{yz}^{d}}{\partial y} + \frac{\partial q_{zz}^{d}}{\partial z} - \frac{\partial \overline{Q}^{d}}{\partial z}\right)m - \frac{1}{2}\left(\frac{\partial q_{xy}^{d}}{\partial x} + \frac{\partial q_{yy}^{d}}{\partial y} + \frac{\partial q_{zy}^{d}}{\partial z} - \frac{\partial \overline{Q}^{d}}{\partial y}\right)n = 0$$

$$\tau_{xy}^{s}\ell + \tau_{yy}^{s}m + \tau_{zy}^{s}n - \frac{1}{2}\left(\frac{\partial q_{xz}^{d}}{\partial x} + \frac{\partial q_{yz}^{d}}{\partial y} + \frac{\partial q_{zz}^{d}}{\partial z} - \frac{\partial \overline{Q}^{d}}{\partial z}\right)\ell + \frac{1}{2}\left(\frac{\partial q_{xx}^{d}}{\partial x} + \frac{\partial q_{yx}^{d}}{\partial y} + \frac{\partial q_{zx}^{d}}{\partial z} - \frac{\partial \overline{Q}^{d}}{\partial x}\right)n = 0$$

$$\tau_{xz}^{s}\ell + \tau_{yz}^{s}m + \tau_{zz}^{s}n + \frac{1}{2}\left(\frac{\partial q_{xy}^{d}}{\partial x} + \frac{\partial q_{yy}^{d}}{\partial y} + \frac{\partial q_{zy}^{d}}{\partial z} - \frac{\partial \overline{Q}^{d}}{\partial y}\right)\ell - \frac{1}{2}\left(\frac{\partial q_{xx}^{d}}{\partial x} + \frac{\partial q_{yx}^{d}}{\partial y} + \frac{\partial q_{zx}^{d}}{\partial z} - \frac{\partial \overline{Q}^{d}}{\partial x}\right)m = 0$$

$$q_{xx}^{d}\ell + q_{yx}^{d}m + q_{zx}^{d}n - \overline{Q}^{d}\ell = 0$$

$$q_{xy}^{d}\ell + q_{yy}^{d}m + q_{zy}^{d}n - \overline{Q}^{d}m = 0$$

$$q_{xz}^{d}\ell + q_{yz}^{d}m + q_{zz}^{d}n - \overline{Q}^{d}n = 0 \qquad (30)$$

where ℓ, m, n are direction cosines of unit normal to boundary

$$\overline{Q}^{d} = q_{xx}^{d}\ell^{2} + q_{yy}^{d}m^{2} + q_{zz}^{d}n^{2} + (q_{xy}^{d} + q_{yx}^{d})\ell m + (q_{xz}^{d} + q_{zx}^{d})\ell n + (q_{yz}^{d} + q_{zy}^{d})mn$$

(a) A Dilatational Mode of a Cube

In the case of zero Poisson's ratio, displacement functions which will satisfy equations (8) and (30) are the following:

$$u = A \sin\left(\frac{m\pi}{2a}\right) x \, e^{i\omega t}$$

$$v = A \sin\left(\frac{m\pi}{2a}\right) y \, e^{i\omega t} \tag{31}$$

$$w = A \sin\left(\frac{m\pi}{2a}\right) z \, e^{i\omega t}$$

with m as odd integers. The above functions will describe symmetric mode shapes with respect to the coordinate planes. Asymmetric modes are given by,

$$u = A \cos\left(\frac{m\pi}{2a}\right) x \, e^{i\omega t}$$

$$v = A \cos\left(\frac{m\pi}{2a}\right) y \, e^{i\omega t} \tag{32}$$

$$w = A \cos\left(\frac{m\pi}{2a}\right) z \, e^{i\omega t}$$

with m as even integers.

The frequency equation corresponding to functions (31) and (32) is as follows:

$$\rho\omega^2 = S_{11}\left(\frac{m\pi}{2a}\right)^2 \tag{33}$$

where $m = 1, 2, 3, \ldots$.

As in the two-dimensional case the frequencies of the dilatational modes of the cube are not influenced by couple-stress.

(b) An Equivoluminal Mode of a Cube

Without considering the effect of couple-stress, the equations of motion (8) reduce to the following:

$$S_{11}\frac{\partial^2 u}{\partial x^2} + S_{12}\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right) + \frac{S_{44}}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right) = \rho\frac{\partial^2 u}{\partial t^2}$$

$$S_{11}\frac{\partial^2 v}{\partial y^2} + S_{12}\left(\frac{\partial^2 w}{\partial y\partial z} + \frac{\partial^2 u}{\partial y\partial x}\right) + \frac{S_{44}}{2}\left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial z\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x\partial y}\right) = \rho\frac{\partial^2 v}{\partial t^2}$$

$$S_{11}\frac{\partial^2 w}{\partial z^2} + S_{12}\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 v}{\partial z\partial y}\right) + \frac{S_{44}}{2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial z\partial y}\right) = \rho\frac{\partial^2 w}{\partial t^2} \quad (34)$$

and boundary conditions (30) reduce to:

$$\begin{aligned} \tau^s_{xx}\ell + \tau^s_{yx}m + \tau^s_{zx}n &= 0 \\ \tau^s_{xy}\ell + \tau^s_{yy}m + \tau^s_{zy}n &= 0 \\ \tau^s_{xz}\ell + \tau^s_{yz}m + \tau^s_{zz}n &= 0 \end{aligned} \quad (35)$$

Displacement functions which describe symmetric modes with respect to the coordinate planes and which satisfy equations (34) and (35) are the following:

$$u = \sin(\frac{m\pi}{2a})x[A_3 \cos(\frac{m\pi}{2a})y - A_2 \cos(\frac{m\pi}{2a})z]e^{i\omega t}$$

$$v = \sin(\frac{m\pi}{2a})y[A_1 \cos(\frac{m\pi}{2a})z - A_3 \cos(\frac{m\pi}{2a})x]e^{i\omega t} \tag{36}$$

$$w = \sin(\frac{m\pi}{2a})z[A_2 \cos(\frac{m\pi}{2a})x - A_1 \cos(\frac{m\pi}{2a})y]e^{i\omega t}$$

with m as odd integers. Asymmetric modes are given by

$$u = \cos(\frac{m\pi}{2a})x[A_3 \sin(\frac{m\pi}{2a})y - A_2 \sin(\frac{m\pi}{2a})z]e^{i\omega t}$$

$$v = \cos(\frac{m\pi}{2a})y[A_1 \sin(\frac{m\pi}{2a})z - A_3 \sin(\frac{m\pi}{2a})x]e^{i\omega t} \tag{37}$$

$$w = \cos(\frac{m\pi}{2a})z[A_2 \sin(\frac{m\pi}{2a})x - A_1 \sin(\frac{m\pi}{2a})y]e^{i\omega t}$$

with m as even integers.

Frequency equation corresponding to equations (36) and (37) is seen to be:

$$\rho\omega^2 = (S_{11} - S_{12})(\frac{m\pi}{2a})^2 \tag{38}$$

where $m = 1, 2, 3, \ldots$.

EXPERIMENTAL STUDY OF FREE VIBRATIONS

Using the experimentally determined elastic constants, one can use the solutions of the equations of motion to predict the normal mode shapes and corresponding frequencies of the models. It is thought to be of considerable value for the development of the theory of couple-stresses to have knowledge of the actual vibration characteristics of the physical models. The experiments and results will now be described in some detail.

1. Experimental Method

The experimental models used in the vibration experiments were suspended by flexible strings from rigid steel frames which were bolted to a massive reinforced concrete base $3 \times 3 \times 10$ feet. The freely suspended models will give the stress-free surface conditions for free vibration of solids. Mounting of models and necessary electrical connections to excite the models are shown in Figures 2 and 6.

Experimental method that has been used is similar to the one discussed in a paper by Hoppmann [3]. Normal mode vibrations were excited in the solids by an electromagnet mounted solidly at the appropriate location. Since the experimental models are partly made out of aluminum blocks, it was necessary to attach a laminated iron piece to the model at a suitable point to facilitate the magnetic excitation. The response of the model was picked up by crystal-type phonograph pick-up.

The magnet is excited from an audio-oscillator which feeds through a resonant type RC circuit in order to increase the driving

force. On account of the nature of the magnet, the frequency from the oscillator is one-half the frequency of the driving force of the magnet. The output from the oscillator is connected to one set of plates of a cathode ray oscilloscope and the magnetic drive is connected across the other set of plates. Since the frequency ratio is 2 to 1, a figure eight Lissajous is shown on the screen of the scope. Crossing a nodal surface with the pick-up turns the figure eight Lissajous into a horseshoe shaped figure on one side of the surface and inverts the figure on the opposite side. So by moving the crystal-type pick-up over the models the nodal surfaces and hence the nodal patterns were determined. This is a precise method for the determination of the modal shapes. It is important to determine the mode shapes and their corresponding frequencies simultaneously, especially since the spectrum of frequencies may be closely packed for any elastic body of the nature under investigation.

2. Experiments on Model in Shape of a Slice of an Infinite Prism

The two-dimensional model was considered as a slice of an infinite prism. The model was freely suspended (Figure 2) from the solid frame by means of two flexible strings so that no constraint was introduced. The model was excited so that motion out of the plane of the model was prevented. The vibrations are then approximately those of plane strain.

A schematic diagram of the experimental apparatus with some details of model and electrical connections is shown in Figure 2. The magnet was mounted on the solid frame and was easily movable in any direction. The model was excited at different points and the mode shapes and frequencies were determined.

After determining the mode shapes, an attempt was made to classify them. It is well-known tnat the complete classification of normal mode vibrations for elastic solids has never been accomplished. Some attempts have been made with some degree of success as discussed by Ekstein and Schiffman [4].

On comparing the experimentally determined mode shapes with the theoretical shape functions it is considered that they represent the dilatational, equivoluminal and face-shear types. In the classification of Ekstein and Schiffman [4], the fundamental modes of dilatational, equivoluminal and face-shear type are referred to as the breathing, longitudinal and shear modes, respectively.

The face-shear type modes were the lowest in frequency. The frequencies of the equivoluminal and dilatational modes were rather close to each other. The mode shapes along with the corresponding frequencies are shown in Figures 3, 4 and 5. It was found that the experimental nodal lines shown in Figure 3 were approximately the same as that given by shape functions,

$$\bar{u} = A \sin\left(\frac{m\pi}{2a}\right)y$$

$$\bar{v} = A \sin\left(\frac{m\pi}{2a}\right)x$$

for odd values of m and

$$\bar{u} = A \cos(\frac{m\pi}{2a})y$$

$$\bar{v} = A \cos(\frac{m\pi}{2a})x$$

for even values of m . Accordingly, the mode shapes are sketched from these functions. Calculated and experimental frequencies are tabulated in Table II. The average density, $\rho_a = \frac{.401 \times 10^{-4}}{h} \frac{\text{lbs.sec.}^2}{\text{in.}^4}$, of the model was used in calculation.

3. Experiments on Three-Dimensional Model

The model was freely suspended from a solid frame by means of three inextensible strings. The method of exciting the model was similar to that discussed previously. A photograph of the apparatus is shown in Figure 6.

The frequencies and mode shapes were determined by exciting the model at different points. An attempt has been made to classify the mode shapes. Besides the analogous shapes of two-dimensional models there were certain modes which were not identified. The classified mode shapes and frequencies are shown in Figures 7, 8 and 9. The unclassified mode shapes and frequencies are given in Figure 10. Theoretical frequencies are calculated on the basis of average density of the model, $\rho_a = \frac{.565 \times 10^{-4}}{h} \frac{\text{lbs.sec.}^2}{\text{in.}^4}$. Calculated and experimental frequencies are shown in Table III.

DISCUSSION AND CONCLUSION

For the models which were studied, it has been found that the wave velocity shows dispersive effects because of couple-stresses and as the wave length becomes indefinitely decreased the phase velocity becomes unbounded. Also, it was found that the velocity of a purely dilatational wave is uninfluenced by the couple rigidity characteristic of the model.

As is well-known, the classification of normal mode vibrations for elastic solids is incomplete. Ekstein and Schiffman [4] have attempted to classify the modes of isotropic cubes by using group theory, but only with a modest degree of success. Classification of the modes for centro-symmetric cubic crystals with or without couple-stress has never been attempted so far as is known. In the present paper some attempt was made to classify the modes on the basis of experimental and theoretical results.

As would be anticipated from analysis of normal mode vibration the dilatational modes are unaffected by the couple rigidity factor. Equi-voluminal and face-shear type modes do show couple-stress effects.

Since the experimental model is inhomogeneous on account of the manner in which its structured form and mass distribution have been designed, the vibration experiments, in relation to the theory, can only be expected to be meaningful for wave lengths which are longer than the size of a typical cell unit. Also, for the higher modes it should be expected that there will be some influence of rotatory inertia of the discrete blocks which does not arise in the theory of continuous media.

From the vibration experiments which were conducted, it was found that the face-shear type modes fall into a lower set of frequencies, the highest of which is about 250 cps, and the dilatational and equi-voluminal modes are in a higher frequency set which starts at about 1100 cps. Also, it was seen that the experimentally determined frequencies of the lower modes check the theoretical values to within 5%. As expected, the frequencies do not check that well for the higher modes for the reason given above.

Unfortunately, since it was not possible to solve the equations of motion for those free vibrations of the models which would be expected to show couple-stress effects, it was then not possible to predict the influence of couple-stress on the frequencies of vibration. However, it is interesting to observe that there was close agreement between experimentally determined modes of vibration and the theoretical modes of vibration which obviously do not involve couple-stresses.

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2. Miller, G. F. and Musgrave, M. J. P., "On the Propagation of Elastic Waves in Aeolotropic Media," III. Media of Cubic Symmetry, Proc. Roy. Soc. A236, p. 352 (1956).

3. Hoppmann, W. H. II, "Flexural Vibrations of Orthogonally Stiffened Cylindrical Shells," Proc. Ninth Intern. Congress of Appl. Mechanics, Bruxelles, 1956.

4. Ekstein, H. and Schiffman, T., "Free Vibrations of Isotropic Cubes and Nearly Cubic Parallelopipeds," Journal of Applied Physics, vol. 27, No. 4, April 1956.

TABLE I

PHASE VELOCITIES OF WAVE PROPAGATION IN VARIOUS DIRECTIONS OF THE PHYSICAL MODEL

<table>
<tr><td colspan="4">[100] Direction: $\ell = 1, \quad m = n = 0$</td></tr>
<tr><td>Displacement</td><td colspan="2">$U \neq 0, \quad V \neq 0, \quad W = 0$</td><td>$U = 0, \quad V = 0, \quad W \neq 0$</td></tr>
<tr><td rowspan="4">Velocity</td><td colspan="2">$C_1 = \frac{1}{12}\sqrt{\frac{S_{11}}{\rho}}$
$= 5310$ ft./sec.</td><td rowspan="4">$C_3 = \frac{1}{12}\sqrt{\frac{1}{\rho}\left(\frac{S_{44}}{2} + \frac{C\pi^2}{\lambda^2}\right)}$
$C_3 = C_2$</td></tr>
<tr><td colspan="2">$C_2 = \frac{1}{12}\sqrt{\frac{1}{\rho}\left(\frac{S_{44}}{2} + \frac{C\pi^2}{\lambda^2}\right)}$</td></tr>
<tr><td>$S_{44} \neq 0, \quad C = 0$</td><td>$S_{44} \neq 0, \quad C \neq 0$</td></tr>
<tr><td>471 ft./sec.</td><td>$\lambda = 3''$
4270 ft./sec.</td></tr>
</table>

<table>
<tr><td colspan="6">[110] Direction: $\ell = \frac{1}{\sqrt{2}}, \quad m = \frac{1}{\sqrt{2}}, \quad n = 0$</td></tr>
<tr><td>Displacement</td><td colspan="2">$U \neq 0, \quad V \neq 0, \quad W = 0$</td><td colspan="3">$U = 0, \quad V = 0, \quad W \neq 0$</td></tr>
<tr><td rowspan="4">Velocity</td><td colspan="2">$C_1 = \frac{1}{12}\sqrt{\frac{1}{\rho}\left(\frac{S_{11} + S_{44}}{2}\right)}$
$= 3780$ ft./sec.</td><td colspan="3" rowspan="2">$C_3 = \frac{1}{12}\sqrt{\frac{1}{\rho}\left(\frac{S_{44}}{2} + \frac{C\pi^2}{2\lambda^2} + \frac{3A}{4}\frac{\pi^2}{\lambda^2}\right)}$</td></tr>
<tr><td colspan="2">$C_2 = \frac{1}{12}\sqrt{\frac{1}{2\rho}\left(S_{11} + \frac{2C\pi^2}{\lambda^2}\right)}$</td></tr>
<tr><td>$C = 0$
$S_{11} \neq 0$</td><td>$C \neq 0$
$S_{11} \neq 0$</td><td>$C = A = 0$
$S_{44} \neq 0$</td><td>$C \neq 0, \; A = 0$
$S_{44} \neq 0$</td><td>$C \neq 0, \; A \neq 0$
$S_{44} \neq 0$</td></tr>
<tr><td>3760 ft./sec.</td><td>$\lambda = 3''$
5680 ft./sec.</td><td>471 ft./sec.</td><td>$\lambda = 3''$
3050 ft./sec.</td><td>$\lambda = 3''$
3060 ft./sec.</td></tr>
</table>

<table>
<tr><td colspan="8">[111] Direction: $\ell = m = n = \frac{1}{\sqrt{3}}$</td></tr>
<tr><td>Displacement</td><td colspan="6">$U = V \neq 0, \quad W \neq 0$</td><td>$U = -V, \quad W = 0$</td></tr>
<tr><td rowspan="6">Velocity</td><td colspan="6">$C_1 = \frac{1}{12}\sqrt{\frac{1}{\rho}\left(\frac{S_{11}}{3} + \frac{2}{3}S_{44} + \frac{2}{3}\frac{A\pi^2}{\lambda^2}\right)}$</td><td rowspan="6">$C_3 = \frac{1}{12}\sqrt{\frac{1}{\rho}\left(\frac{S_{11}}{3} + \frac{S_{44}}{6} + \frac{2C\pi^2}{9\lambda^2} + \frac{5A\pi^2}{6\lambda^2}\right)}$
$C_3 = C_2$</td></tr>
<tr><td colspan="3">$S_{11} \neq 0, \; S_{44} \neq 0, \; A=0$</td><td colspan="3">$S_{11} \neq 0, \; S_{44} \neq 0, \; A \neq 0$</td></tr>
<tr><td colspan="3">3110 ft./sec.</td><td colspan="3">$\lambda = 3''$
3120 ft./sec.</td></tr>
<tr><td colspan="6">$C_2 = \frac{1}{12}\sqrt{\frac{1}{\rho}\left(\frac{S_{11}}{3} + \frac{S_{44}}{6} + \frac{2C\pi^2}{9\lambda^2} + \frac{5A\pi^2}{6\lambda^2}\right)}$</td></tr>
<tr><td colspan="2">$S_{11} \neq 0, S_{44} \neq 0,$
$C = A = 0$</td><td colspan="2">$S_{11} \neq 0, S_{44} \neq 0,$
$C \neq 0, \; A = 0$</td><td colspan="2">$S_{11} \neq 0, S_{44} \neq 0,$
$C \neq 0, \; A = 0$</td></tr>
<tr><td colspan="2">3080 ft./sec.</td><td colspan="2">$\lambda = 3''$
3680 ft./sec.</td><td colspan="2">$\lambda = 3''$
3690 ft./sec.</td></tr>
</table>

TABLE II

FREQUENCIES OF TWO-DIMENSIONAL MODEL

(a) A Dilatational Mode

m	Frequency (cps)	
	Experimental	Theoretical
1	1150	1195
2	1880	2390
3	2900	3585
4	3600	4780

(b) An Equivoluminal Mode

m	Frequency (cps)	
	Experimental	Theoretical
1	1110	1195
2	1960	2390
3	2740	3585
4	3450	4780

TABLE III

FREQUENCIES OF THREE-DIMENSIONAL MODEL

(a) A Dilatational Mode

m	Frequency (cps)	
	Experimental	Theoretical
1	1520	1610
2	2800	3220

(b) An Equivoluminal Mode

m	Frequency (cps)	
	Experimental	Theoretical
1	1460	1610
2	2640	3220

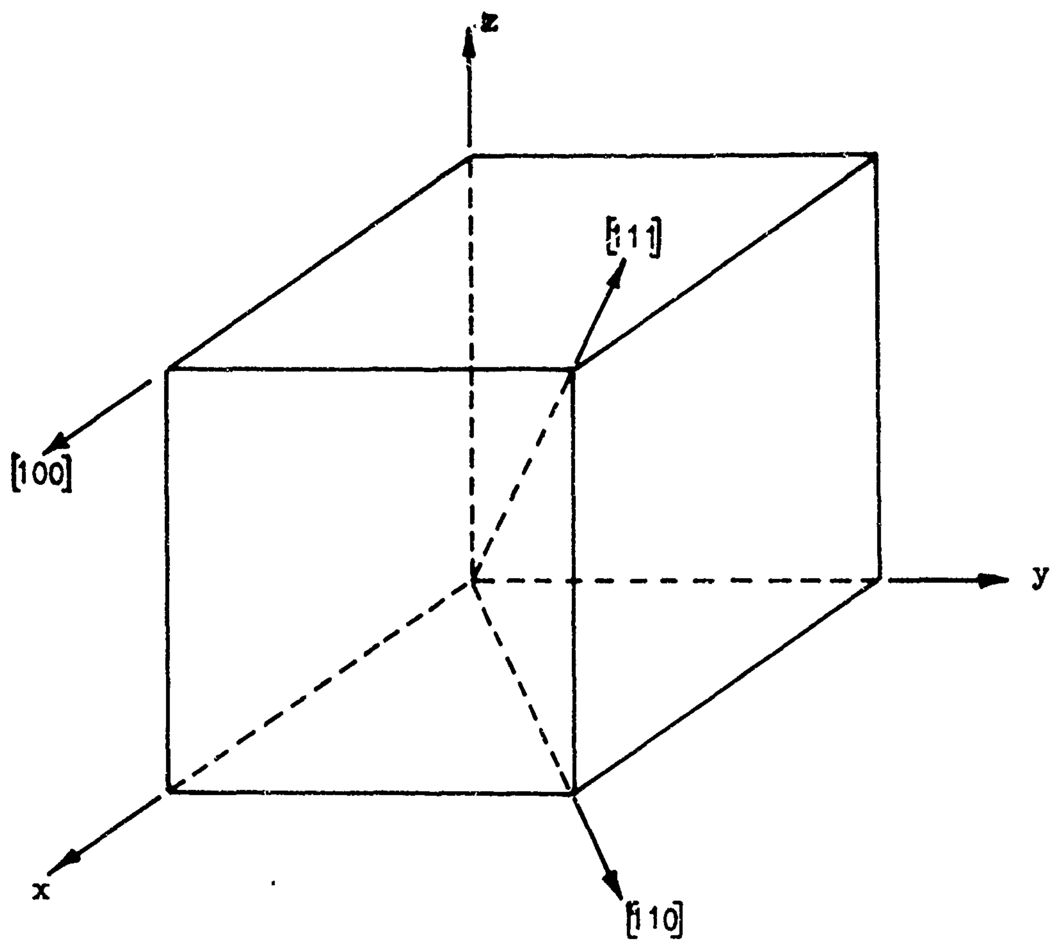

FIGURE 1. Various Directions of Wave Normal

Considered in the Calculation of Phase Velocities

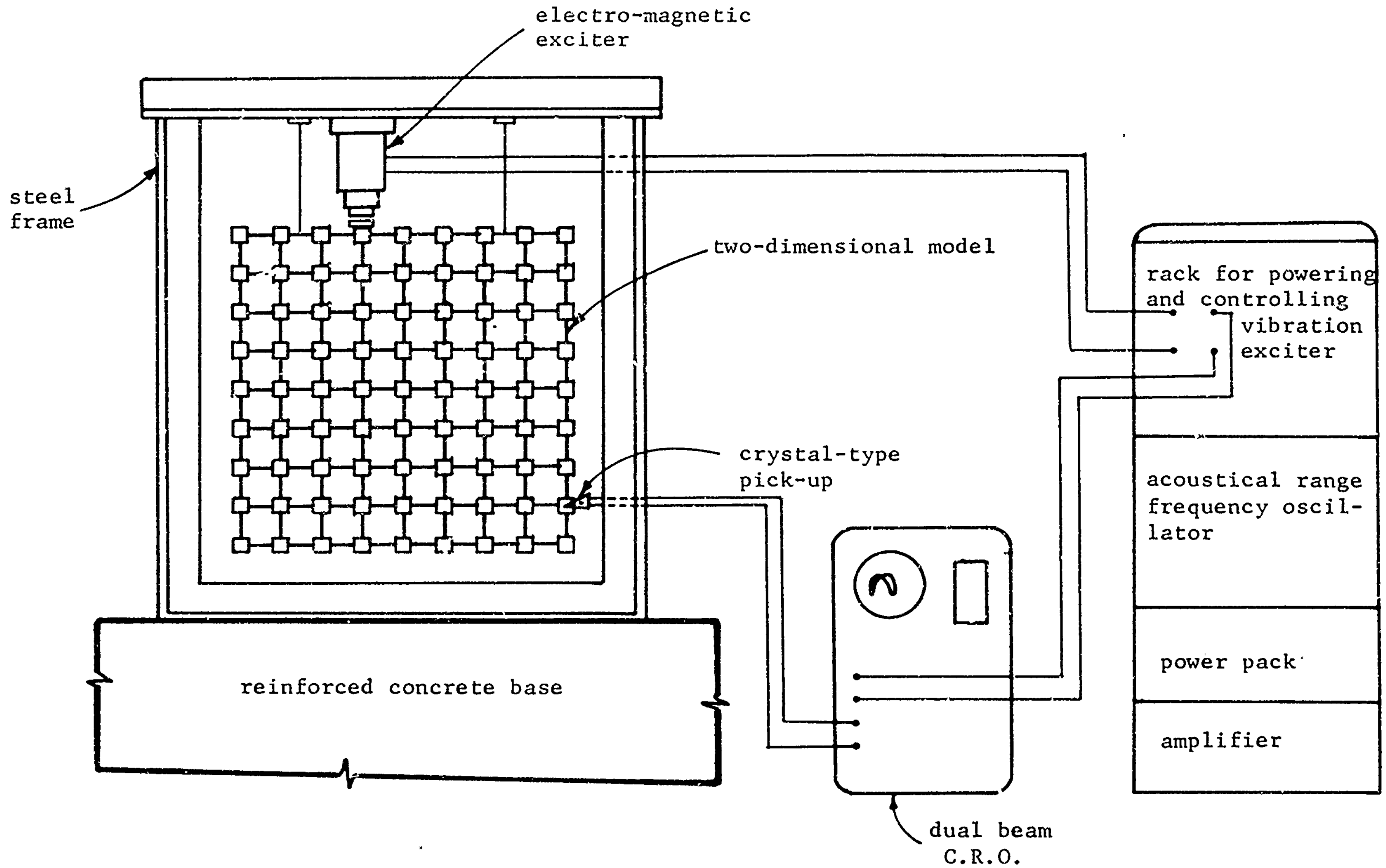

Figure 2. APPARATUS FOR VIBRATION OF TWO-DIMENSIONAL MODEL

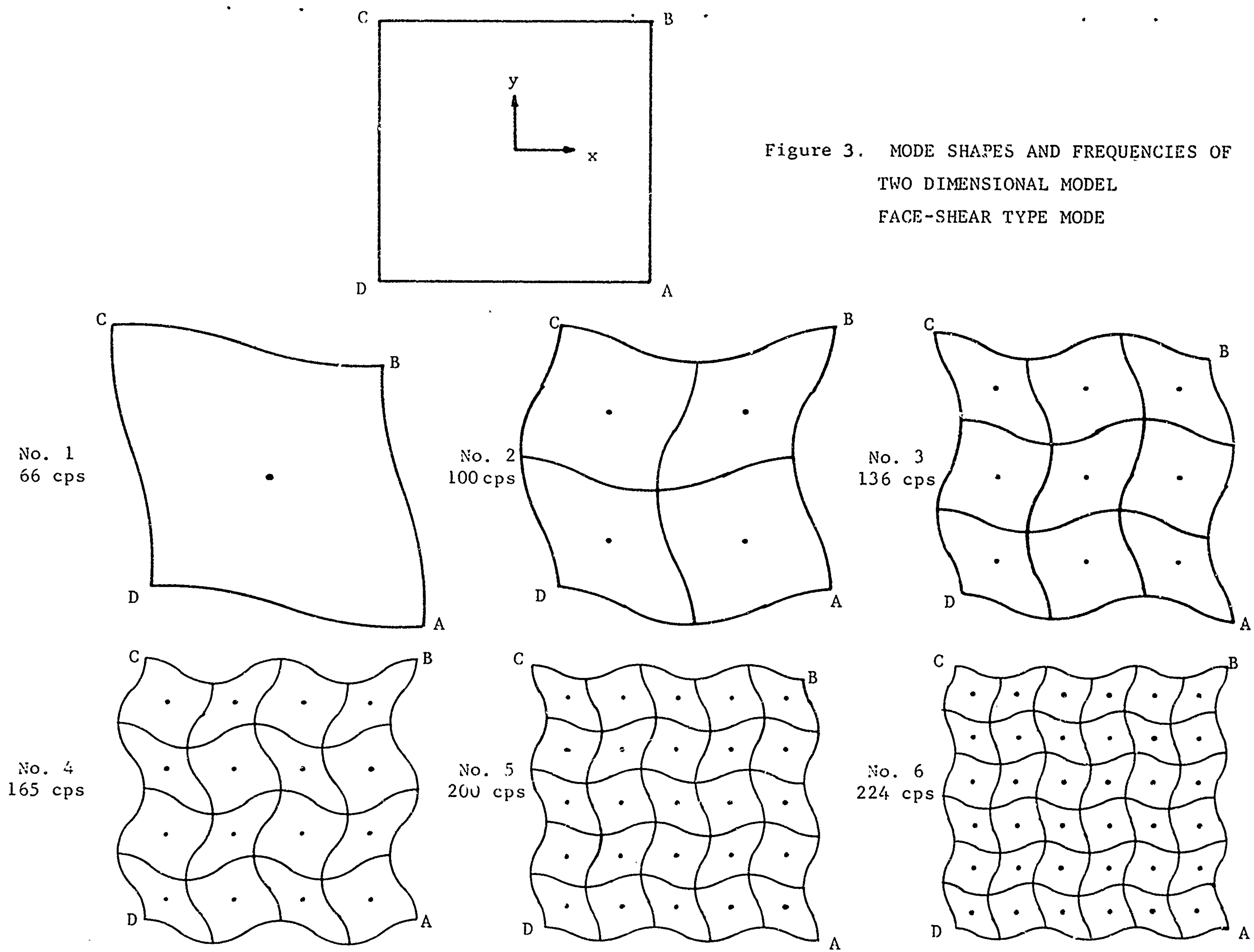

Figure 3. MODE SHAPES AND FREQUENCIES OF TWO DIMENSIONAL MODEL FACE-SHEAR TYPE MODE

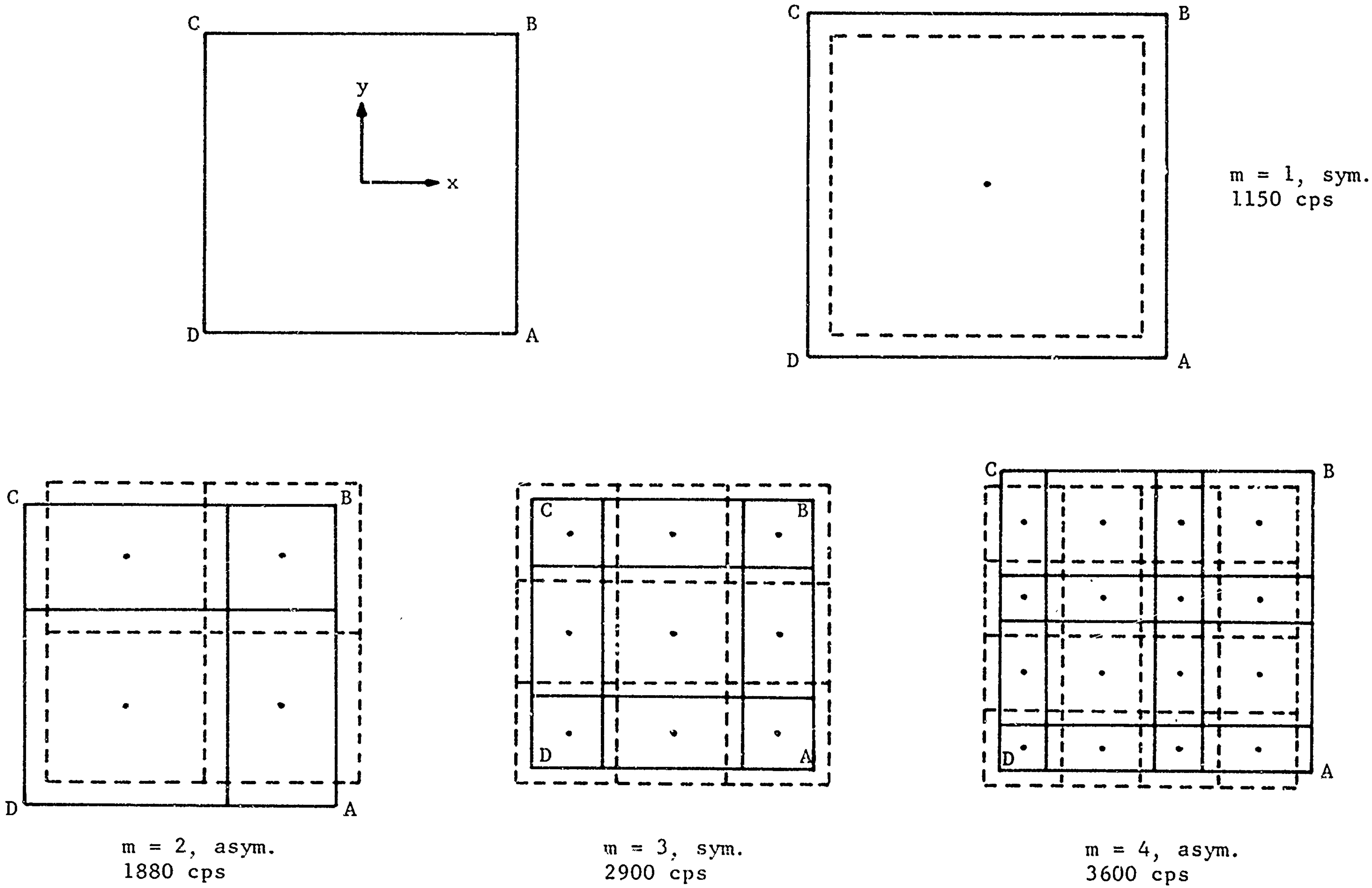

Figure 4. MODE SHAPES AND FREQUENCIES OF TWO DIMENSIONAL MODEL
A DILATATIONAL MODE

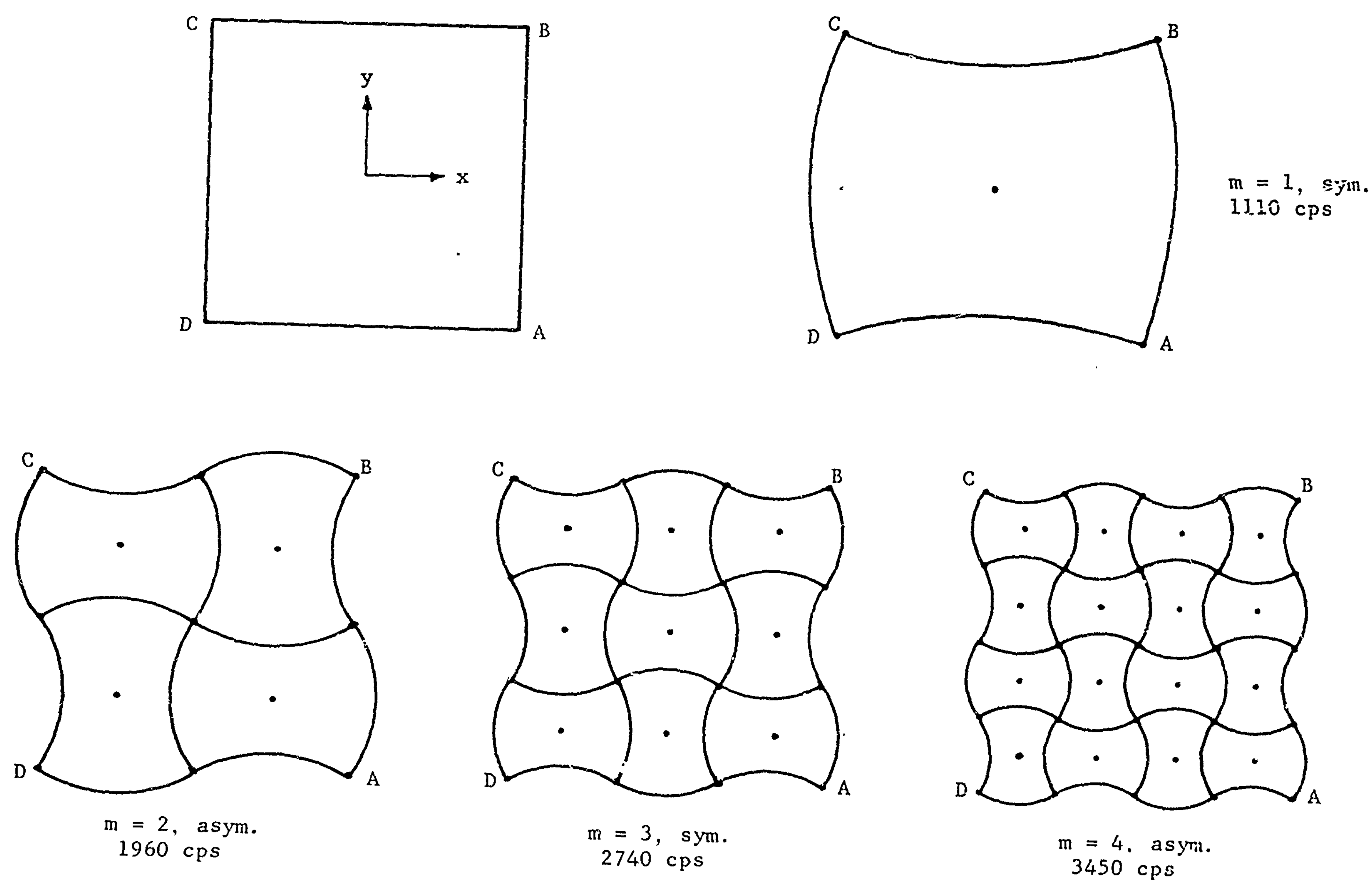

Figure 5. MODE SHAPES AND FREQUENCIES OF TWO-DIMENSIONAL MODEL AN EQUIVOLUMINAL MODE

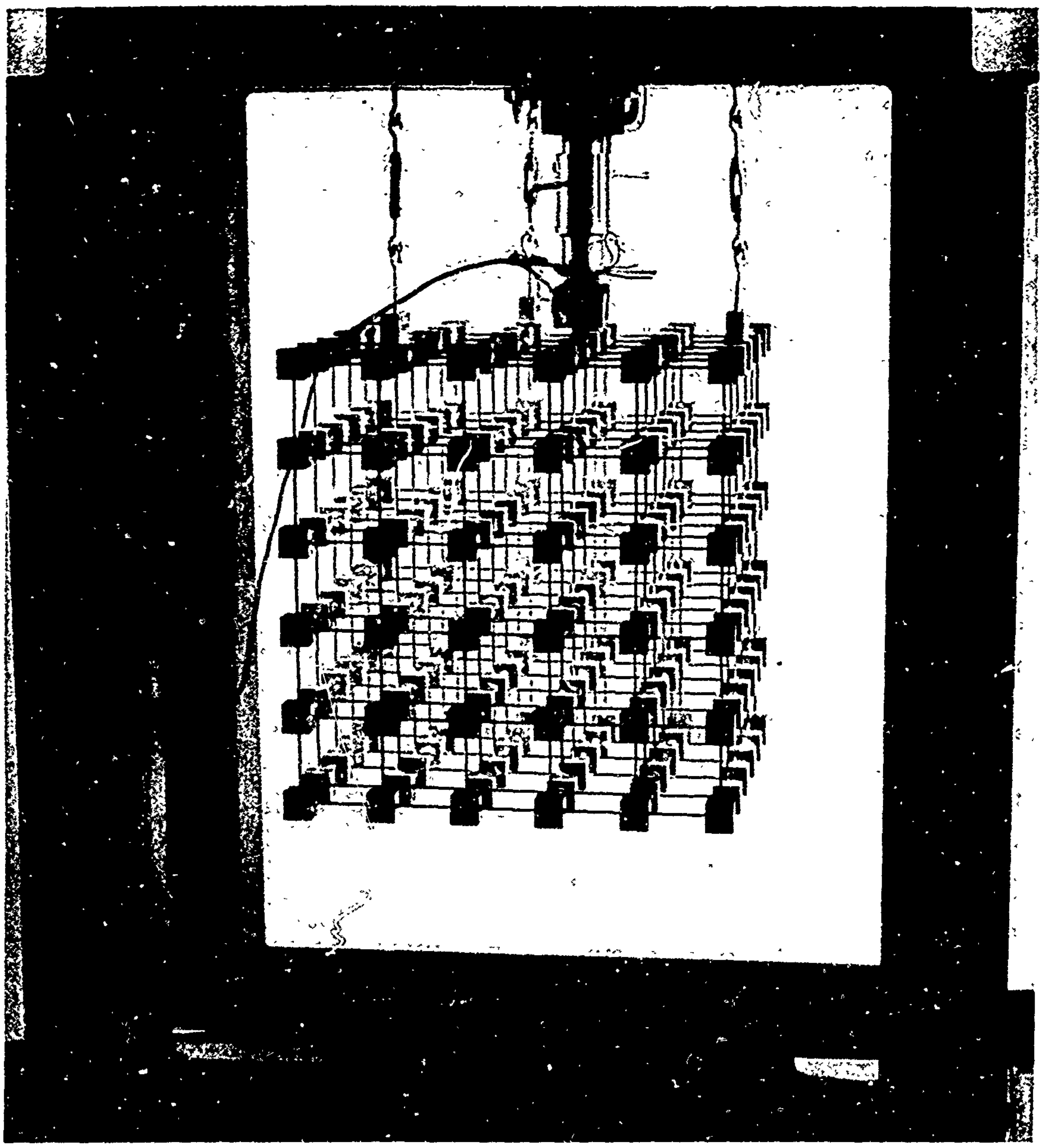

Figure 6. APPARATUS FOR VIBRATION OF THREE-DIMENSIONAL MODEL

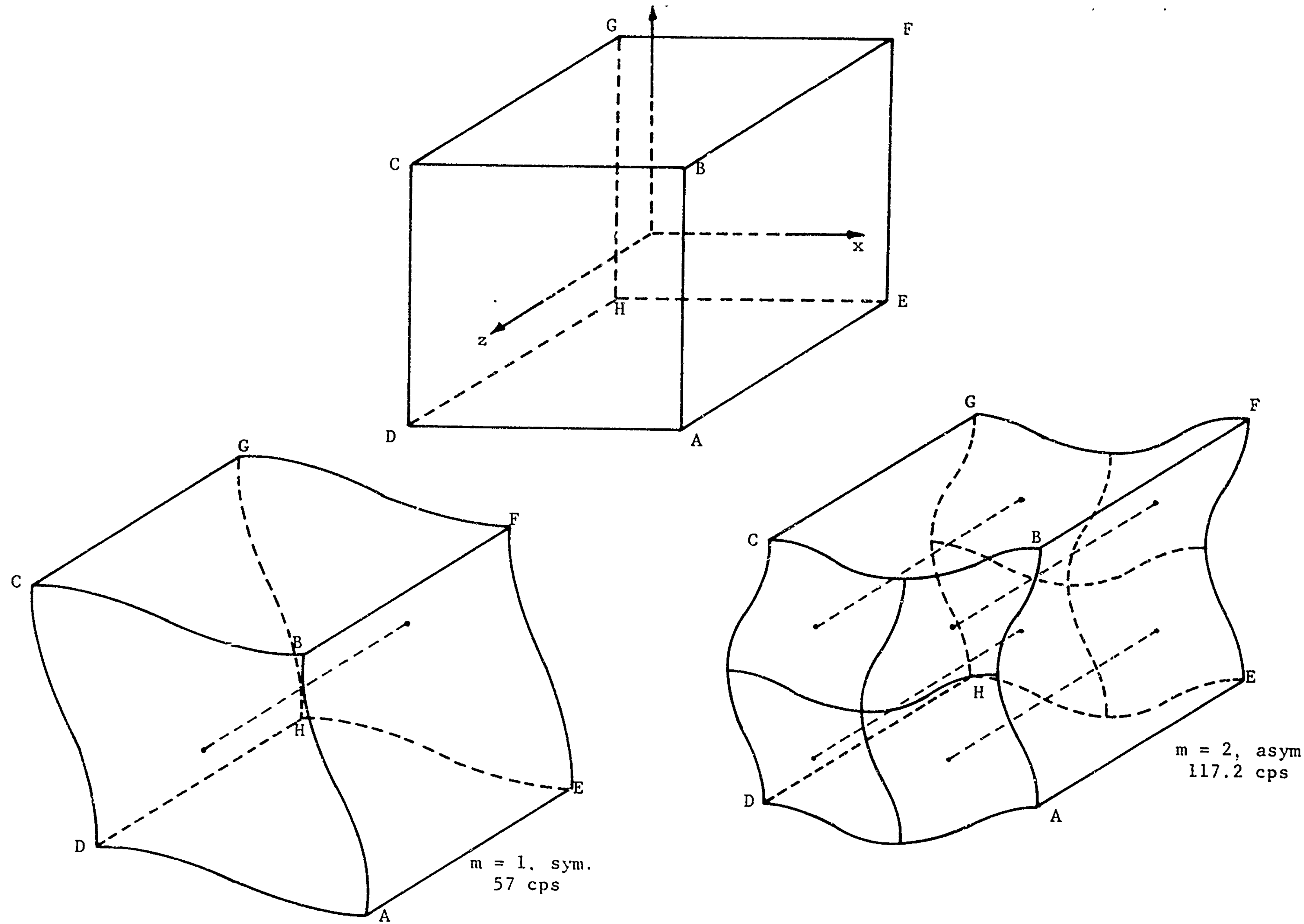

Figure 7. MODE SHAPES AND FREQUENCIES OF THREE-DIMENSIONAL MODEL FACE-SHEAR TYPE MODE

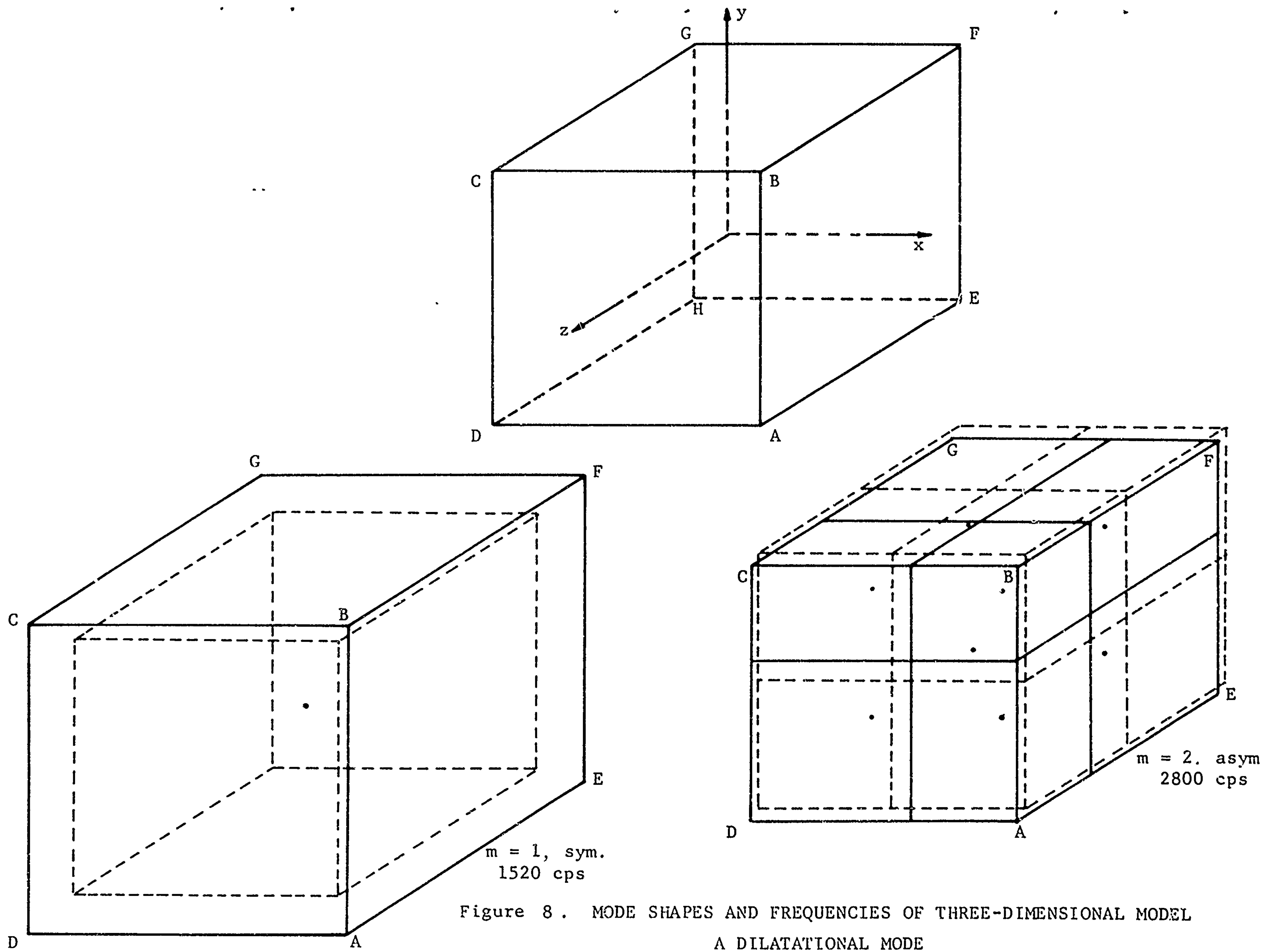

Figure 8 . MODE SHAPES AND FREQUENCIES OF THREE-DIMENSIONAL MODEL
A DILATATIONAL MODE

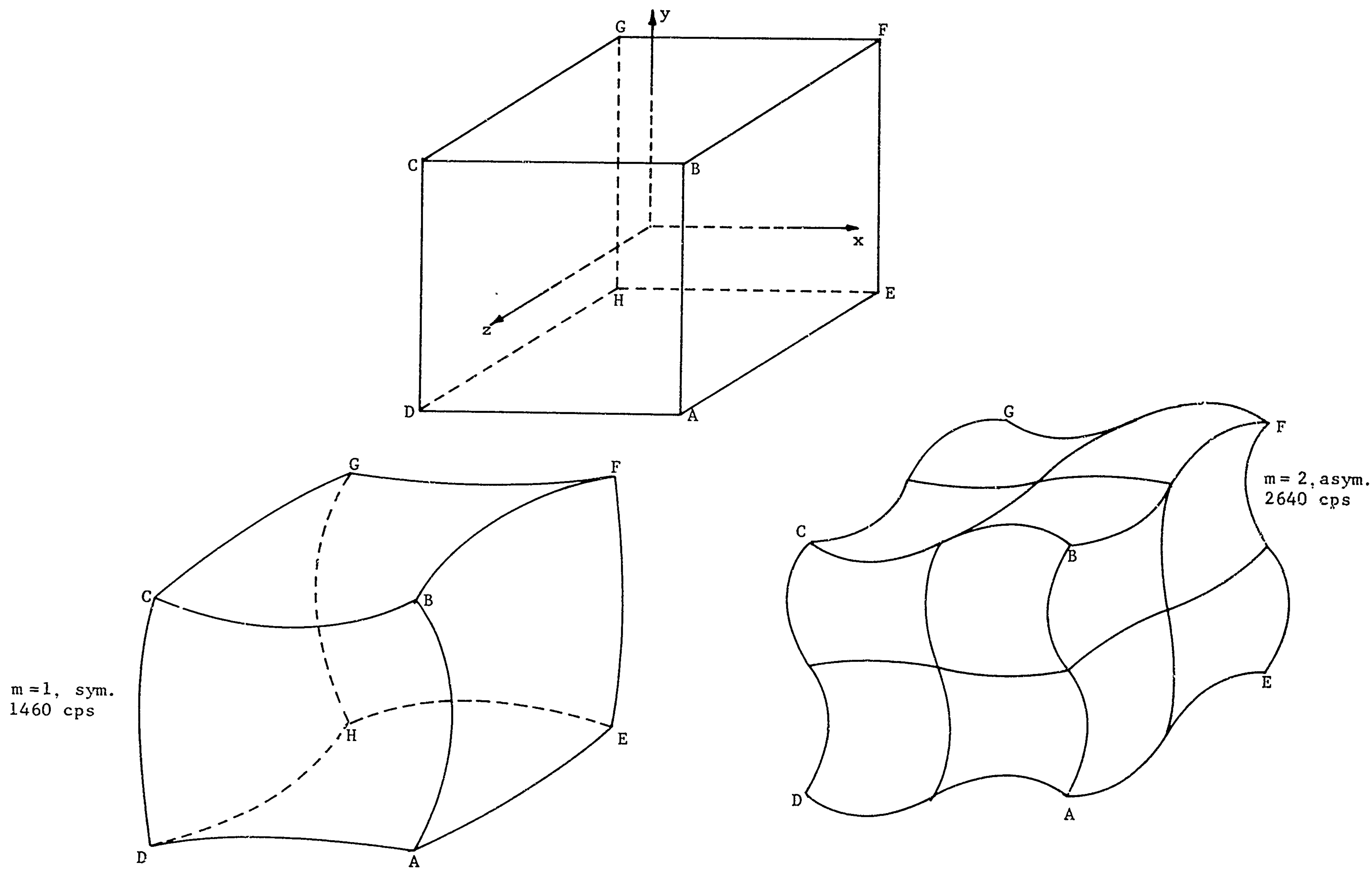

Figure 9. MODE SHAPES AND FREQUENCIES OF THREE-DIMENSIONAL MODEL AN EQUIVOLUMINAL MODE

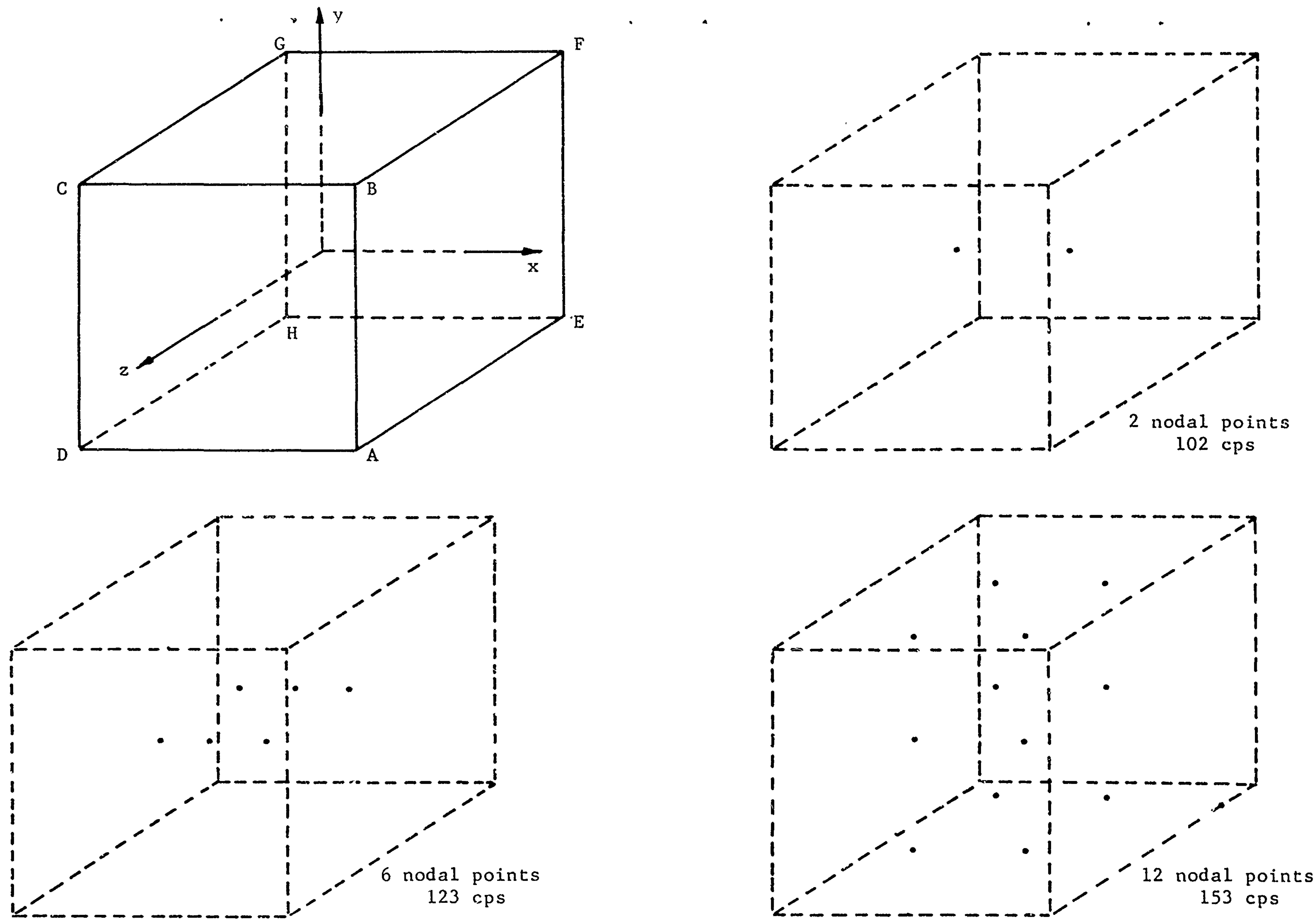

Figure 10. NODAL POINTS AND FREQUENCIES OF THREE-DIMENSIONAL MODEL

UNCLASSIFIED MODES